

Limits of Functions & Continuity

ISI & CMI Crash Course 2024

lec 1

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⊛ Limits of Functions.

(we will be working solely in $\mathbb{R}$, & will ignore topological technicalities such as limit points, etc)

⊙ Introduction

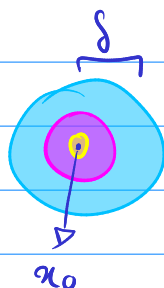
- Defn: Suppose $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, $A \ni x_0$.
If $\exists L \in \mathbb{R} \mid \forall \varepsilon > 0, \exists \delta > 0 \mid$
 $0 < |x - x_0| < \delta, x \in A \Rightarrow |f(x) - L| < \varepsilon$

... then we say " $f(x)$ has limit L @ x_0 ", written as:
 $\lim_{x \rightarrow x_0} f(x) = L$ (or) $f(x) \rightarrow L$ as $x \rightarrow x_0$

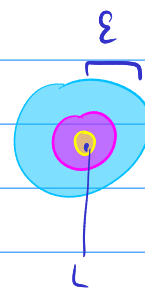
You can get $f(x)$ arbitrarily close to L simply by moving x closer to x_0 .

Note that we are not commenting on $f(x_0)$ itself. $\therefore 0 < |x - x_0|$
So we will come to this during continuity. metric closure

- Equivalently ... $\bigcap_{\delta > 0} f[(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)] = \{L\}$
 $\Leftrightarrow \lim_{x \rightarrow x_0} f(x) = L$ ↗ $B(x_0, \delta) \setminus \{x_0\}$



$\xrightarrow{\quad}$
 f -image.



• Eg: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Proof: Fix $\epsilon > 0$. Want to find $\delta > 0 \mid \forall x \in (-\delta, 0) \cup (0, \delta)$

(circular arc & triangle)
proof

$$\left| \frac{\sin x}{x} - 1 \right| < \epsilon$$

$$\Rightarrow \frac{\sin x}{x} \in (1 \pm \epsilon)$$

<https://math.stackexchange.com/questions/75130/how-to-prove-that-lim-limits-x-to-0-frac-sin-x-x-1>

• Sequential Defn of Limits.

$$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, A \ni x_0$$

$$\lim_{x \rightarrow x_0} f(x) = L \in \mathbb{R}$$

$$\forall \{x_n\} \subseteq A \setminus \{x_0\}$$

$$\Leftrightarrow x_n \rightarrow x_0$$

$$\Rightarrow f(x_n) \rightarrow L$$

• Algebra of Limits. (assuming these limits exist)

(i) $\forall \lambda \in \mathbb{R} \quad \lim_{x \rightarrow x_0} [f(x) + \lambda g(x)] = \lim_{x \rightarrow x_0} f(x) + \lambda \lim_{x \rightarrow x_0} g(x)$

(ii) $\lim_{x \rightarrow x_0} f(x)g(x) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$

(iii) If g is non-vanishing around x_0 , $\lim_{x \rightarrow x_0} g(x) \neq 0$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \left(\lim_{x \rightarrow x_0} f(x) \right) / \left(\lim_{x \rightarrow x_0} g(x) \right)$$

• If $f \leq g$ around x_0 , $\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$

↳ Corollary (Sandwich Theorem) $f \leq h \leq g$, $f, g \rightarrow L$
 $\Rightarrow h \rightarrow L$

⊙ Some Standard Limits.

$$i) \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$$

$$ii) \quad \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} =: e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

$$iii) \quad \lim_{x \rightarrow 0} \frac{(1+x)^a - 1}{x} = a \quad \forall a \in \mathbb{R}$$

$$iv) \quad \lim_{x \rightarrow 0} \frac{(\ln(1+x))}{x} = 1 = \lim_{x \rightarrow 0} \frac{(e^x - 1)}{x}$$

← inverses of each other →

$$v) \quad \lim_{x \rightarrow \pm\infty} \frac{P(x)}{Q(x)} = \begin{cases} 0 & \text{if } \deg Q > \deg P \\ \text{ratio of leading coeffs} & \text{if } \deg Q = \deg P \\ \pm\infty & \text{if } \deg Q < \deg P \end{cases}$$

polynomials.

⊙ Divergence Criteria

i) Unbounded f .

ii) 2 sequences converge to different values.

iii) Infinite oscillations! $\hookrightarrow \sin\left(\frac{1}{x}\right)$ @ 0.

No limit exists.

⊛ Extensions to Functional Limits.

⊙ Infinite Limits.

- Suppose $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$. $A \ni x_0$. If $\forall M \in \mathbb{R}$, $\exists \delta > 0 \mid 0 < |x - x_0| < \delta \Rightarrow f(x) > M$ then we say: $f(x) \rightarrow \infty$ as $x \rightarrow x_0$. or $\lim_{x \rightarrow x_0} f(x) = \infty$

- (similarly for $-\infty$)

- Eg.: $\frac{1}{x^2} \rightarrow \infty$ as $x \rightarrow 0$

⊙ Limits @ $\pm\infty$

- Suppose $f: [a, \infty) \rightarrow \mathbb{R}$. If $\exists L \in \mathbb{R} \mid \forall \epsilon > 0 \exists x^* \geq a \mid x \geq x^* \Rightarrow |f(x) - L| < \epsilon$. then we say $f(x) \rightarrow L$ as $x \rightarrow \infty$ or $\lim_{x \rightarrow \infty} f(x) = L$.

- (similar for $-\infty$)

- Eg.: $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$

⊙ One-sided Limits.

- Suppose $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$. $x_0 \in A$. We define the left-hand limit (LHL) as follows: If $\exists L \in \mathbb{R} \mid \forall \epsilon > 0, \exists \delta > 0 \mid 0 < x_0 - x < \delta, x \in A, |f(x) - L| < \epsilon$ then we say $f(x) \rightarrow L$ as $x \rightarrow x_0^-$ or $\lim_{x \rightarrow x_0^-} f(x) = L$

- (similar for RHL: $f(x) \rightarrow L$ as $x \rightarrow x_0^+$ or $\lim_{x \rightarrow x_0^+} f(x) = L$)
 $0 < x - x_0 < \delta$

- NOTE: limit exist iff LHL, RHL exist & are equal.

- $\epsilon\delta$: $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$, $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

- If $f(x) \rightarrow L$, & $\forall \{x_n\} \mid x_n \rightarrow x_0$, $f(x_n) < L$

then we can say $f(x) \rightarrow L^-$ or if $f(x_n)$ monotone,
 $f(x) \uparrow L$

- (similarly for $f(x) \rightarrow L^+$ or $f(x) \downarrow L$)

⑤ Continuity

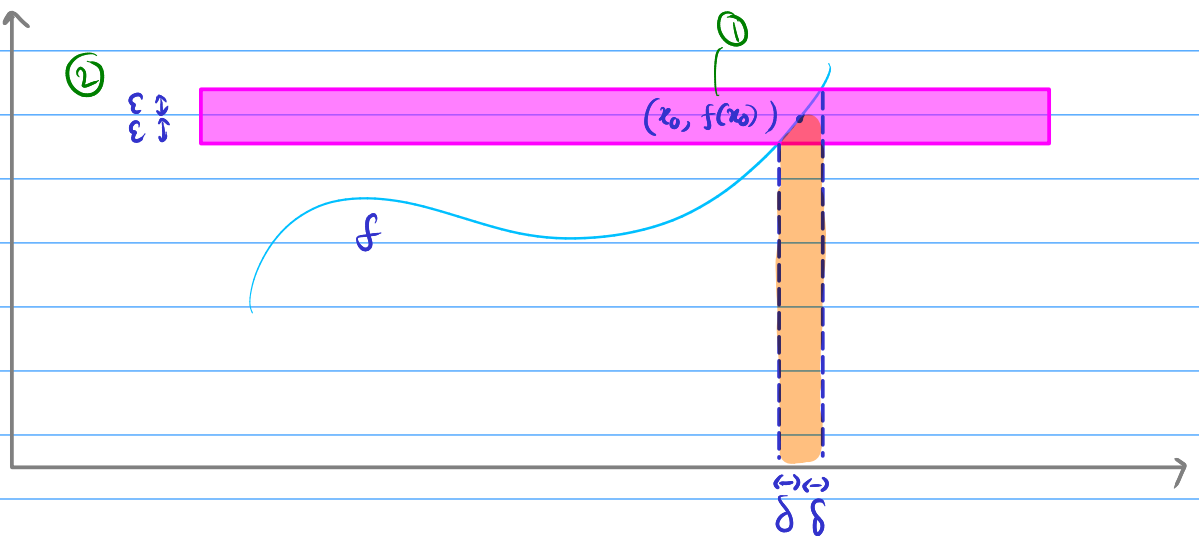
⊙ Introduction

- Defⁿ: We say $f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous @ $x_0 \in A$
iff $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

$$\Leftrightarrow \forall \{x_n\} \subseteq A \setminus \{x_0\} \mid x_n \rightarrow x_0 \Rightarrow f(x_n) \rightarrow f(x_0)$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \mid |x - x_0| < \delta, x \in A \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

- Illustration:



- In practice the definition just says we can get as close as needed to the output @ input x_0 by moving closer to input x_0 .

$$x_n \rightarrow x_0 \iff \begin{cases} f(x_n) \rightarrow f(x_0) \\ f \text{ is continuous @ } x_0. \end{cases}$$

Essentially, there are no "holes" or "gaps" in the graph.

② Algebra of Continuous Functions

- If $f, g : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ are continuous @ $x_0 \in A$.
 $\forall \lambda \in \mathbb{R}$, $f + \lambda g$ is continuous @ $x_0 \in A$.
 fg is continuous @ $x_0 \in A$.
- If $f, g : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ are continuous @ $x_0 \in A$
 & g is non-vanishing in a neighbourhood of x_0
 then f/g is continuous @ x_0 . (in A)
- If $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous @ $x_0 \in A$.
 $g : f(A) \rightarrow \mathbb{R}$ is continuous @ $f(x_0)$
Then: $g \circ f$ is " " @ x_0

(Hw) Prove ↑ using the definition.

② Some Standard Functions

continuous on $S \subseteq A = \text{Dom}(f)$
 $\Rightarrow$ continuous @ $x \forall x \in S$

- All polynomials are continuous on $\mathbb{R}$
 $(\because 1 \text{ \& Id are continuous } \Rightarrow \mathbb{R}[x] \text{ is a subalgebra of } C^0(\mathbb{R}))$

$\downarrow$
 $f: x \mapsto 1$

$\downarrow$
 $f: x \mapsto x$

$\downarrow$ generated by 1 & Id
 polynomials in x
 w/ real coeffs.

$\downarrow$
 real continuous
 functions.
- All rational functions of the form P/Q are continuous
 on $\mathbb{R} \setminus \{x \mid Q(x) = 0\}$.
- sin, cos are cts. on $\mathbb{R}$. (Rest you can derive)

• Dirichlet Function / Indicators

$$\mathbb{1}_{\mathbb{Q}} : x \mapsto \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

The above f^n is discontinuous everywhere.

Proof: $\mathbb{Q}, \mathbb{Q}^c$ are dense in $\mathbb{R}$



↳ they intersect w/ every open set (open interval)

Essentially, you can find a rational & irrational no. b/w any two distinct real numbers.

Take $\varepsilon = \frac{1}{2}$. Suppose f is cts @ x_0 .

$$\Rightarrow \exists \delta > 0 \mid \forall x \in (x_0 \pm \delta), \quad |f(x) - f(x_0)| < \varepsilon = \frac{1}{2}$$

$$\Rightarrow \forall x \in (x_0 \pm \delta), \quad f(x) = f(x_0). \quad \textcircled{1}$$

$$\text{But, } \forall \delta > 0, \exists r, s \in (x_0 \pm \delta) \mid r \in \mathbb{Q}, s \in \mathbb{Q}^c \\ \Rightarrow f(r) \neq f(s)$$

$\therefore f$ can't be constant on $(x_0 \pm \delta)$. ②

① & ② contradict each other. ■

• Thomae's Function

$$T(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \setminus \{0\} \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This function is continuous on $\mathbb{R} \setminus \mathbb{Q}$

Proof: (Ref: Bartle-Shurbaert pp. 128)

a) By density of $\mathbb{Q}^c$ in $\mathbb{R}$, $\forall x_0 \in \mathbb{Q}$,
 $\exists \{x_n\} \subseteq \mathbb{Q}^c \mid x_n \rightarrow x_0 \Rightarrow f$ is discontinuous @ x_0

But, $f(x_n) \equiv 0$, $f(x_0) > 0 \Rightarrow f(x_n) \not\rightarrow f(x_0)$

b) TSI: f is continuous on $\mathbb{Q}^c$.

Fix any $x_0 \in \mathbb{Q}^c$, $M > 0$.

Define: $\mathbb{Q}_M = \{x_n = p_n/q_n \in \mathbb{Q} \mid q_n \leq M\}$

Observe, $(x_0 \pm 1) \cap \mathbb{Q}_M$ is a finite set.

$\Rightarrow$ can choose $\varepsilon > 0$ small $\mid (x_0 \pm \varepsilon) \cap \mathbb{Q}_M = \emptyset$
 ($\because x_0 \notin \mathbb{Q}$).

$\Rightarrow \{x_n = \frac{p_n}{q_n}\}_{n \in \mathbb{N}} \subseteq \mathbb{Q} \mid x_n \rightarrow x_0$

{ Justify that
 checking only
 rational seqⁿ.
 suffices }

$\Rightarrow q_n \rightarrow \infty$

$\Rightarrow 1/q_n \rightarrow 0$

$\Rightarrow f(x_n) \rightarrow 0 = f(x_0)$

(new)

⑥ EVT, IVT & Interval Mapping Property

- Result: S closed & bounded in $\mathbb{R}$, $f: S \rightarrow \mathbb{R}$ cts.
 $\Rightarrow f(S) \subseteq \mathbb{R}$ bounded.

→ Proof: Suppose $f(S)$ not bounded. (element-wise)
 $\Rightarrow$ (can find $\{x_n\} \subseteq S \mid \{f(x_n)\}_{n \in \mathbb{N}} \geq \{n\}_{n \in \mathbb{N}}$)

Now, $\{x_n\}$ bounded $\Rightarrow \exists \{x_{n_k}\}_{k \in \mathbb{N}}$ convergent by Bolzano-Weierstrass.

Suppose $x_{n_k} \rightarrow x_0$. $\because S$ closed, $x_0 \in S$.
 $\Rightarrow f(x_0) \in f(S)$

By continuity of f , $f(x_{n_k}) \rightarrow f(x_0)$.

But, $\{f(x_{n_k})\}_{k \in \mathbb{N}} \subseteq \{f(x_n)\}_{n \in \mathbb{N}}$ viz. unbounded

$\Rightarrow \{f(x_{n_k})\}$ is unbounded. $\Rightarrow \Leftarrow$. 

→ Thus we have: for any real-valued continuous function on a compact set, the range is bounded:

$$\exists m, M \in \mathbb{R} \mid \forall x \in S \text{ compact, } m < f(x) < M$$

However, something stronger holds!

(in $\mathbb{R}^k$, compact = closed & bounded)
C Heine-Borel

• Extremal Value Theorem

$\forall S$ closed & bounded, $f: S \rightarrow \mathbb{R}$ continuous
 $\exists m, M \in \mathbb{R} \mid \forall x \in S, m \leq f(x) \leq M.$
& $\exists x_{\min}, x_{\max} \in S \mid f(x_{\min}) = m \text{ \& } f(x_{\max}) = M.$

→ Proof: $M = \sup f(S)$ (very similar for $m = \inf f(S)$)

Now, $\because \mathbb{R}$ has sup property,

$\forall n, M - \frac{1}{n}$ is not an upper bound of $f(S)$

$\Rightarrow \forall n, \exists x_n \in S \mid M - \frac{1}{n} < f(x_n) \leq M.$

Now, $f(x_n) \rightarrow M$, also, $\{x_n\}$ is bounded.

By Bolzano-Weierstrass, $\exists \{x_{n_r}\} \mid x_{n_r} \rightarrow x$
 $\because S$ closed, $S \ni x \Rightarrow x_{n_r} \rightarrow x$ in S

By continuity of f : $f(x_{n_r}) \rightarrow f(x)$

But, $f(x_{n_r}) \rightarrow M \Rightarrow \underline{\underline{f(x) = M}}$

$\therefore$ Found $x_{\max} := x$ (similar for $x_{\min}$)

→ Essentially, the above says that the bounds on the range are actually attained at some point.

However, there is an even stronger statement!

Intermediate Value Property

$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to have intermediate value property iff: $\forall x, y \in A, x < y$

$\forall \xi$ between $f(x), f(y)$, $\exists z \in (x, y) \cap A$ |
 $\in (\min\{f(x), f(y)\}, \max\{f(x), f(y)\})$ $f(z) = \xi$

→ Eg: $x \mapsto e^x$ on $(0, \infty)$ ✓ | $x \mapsto 1/x$ on $(-1, 1)$ ✗
 $x \mapsto x^2$ on $\mathbb{Q}$ ✗
 (Id) $x \mapsto x$ on $\mathbb{Q}$ ✗ | $x \mapsto \ln x$ on $(0, 1)$ ✓
 $x \mapsto [x]$ on $\mathbb{R}$ ✗
 → functions are continuous, but $\mathbb{Q}$ is NOT complete!
 $x \mapsto x$ on $[1, 2] \cup [3, 4]$ ✗
 continuity holds, set is complete, but NOT connected!

Intermediate Value Theorem (won't prove, Ref B-S pp. 137) § 5.3.5-5.3.10 extension, used often

- equivalent {
- (i) IVP holds for a continuous function on a complete connected set in $\mathbb{R}$. (think intervals)
 - (ii) A continuous function maps a compact, complete, connected set in $\mathbb{R}$ to a compact, complete, connected set in $\mathbb{R}$. *
- (target)

(Fact $\Rightarrow$ compact, complete, connected set = intervals) (closed & bounded)

Interval Mapping Property.

★ Theorem [edit]
 The intermediate value theorem states the following:
 Consider an interval $I = [a, b]$ of real numbers $\mathbb{R}$ and a continuous function $f: I \rightarrow \mathbb{R}$. Then

- Version I. if u is a number between $f(a)$ and $f(b)$, that is,
 $\min(f(a), f(b)) < u < \max(f(a), f(b))$,
 then there is a $c \in (a, b)$ such that $f(c) = u$. (implies)
- Version II. the image set $f(I)$ is also an interval (closed), and it contains $[\min(f(a), f(b)), \max(f(a), f(b))]$.